Technical Paper:

Valuation of Equity / FX Instruments

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1 Vanilla Equity Option

1.1 Introduction

A vanilla equity option is a financial instrument with the following properties: The owner of the option has the right (not the obligation) to buy (in the case of a call option) or sell (in the case of a put option) one equity at the option expiry date (in case of a European option), at certain dates during the lifetime of the option (in case of a Bermudan option) or every day during the lifetime of the option (in case of an American option) for a certain amount (the strike price) of money. The underlying equity has a given spot price and may pay discrete dividends at certain dates and may be subject to a continuous equity yield. One is interested in the fair value of such options at the valuation date (a possible trade will be settled at the settlement date) due to a given interest rate curve (yield curve or swap curve) and according to a given volatility curve which describes the random behavior of the underlying equity. For hedging purposes, the Greeks are of interest: delta, gamma (first and second derivative of the option value with respect to the spot price of the underlying), theta (first derivative of the option value with respect to time), vega and volatility convexity (first and second derivative of the option value with respect to volatility), delta vega (mixed derivative with respect to the spot price of the underlying and with respect to the volatility).

1.2 Vanilla Equity Option under Black Scholes model

By the use of the UnRisk PRICING ENGINE the value of a vanilla equity option may be calculated in two ways:

- by the use of the analytic formula (i.e. the Black-Scholes formula) which is valid for the valuation of a European option on a non-dividend paying equity. If the underlying equity pays discrete dividends, it is market practice to discount the dividends back to the valuation date (for discounting the curve (yield curve - equity yield curve) is used), subtract them from the given spot price of the equity and calculate the value of the European option due to this new spot price. The analytic solution for the value of a European call option is given by:

  $$C_t = S_0 * e^{-y(T-t)} * N(d_1) - Xe^{-r(T-t)}N(d_2)$$

  where S_0 is the given spot price of the equity, X is the strike price of the option, y is the continuous forward rate (from t to T) per year due to the given equity yield curve, r is the continuous forward rate (from t to T) per year according to the given yield curve, T is the expiry date of the option, t is the valuation date (T-t has to be expressed in years), N(x) is the cumulative probability distribution for the standard normal distribution at x, i.e.

  $$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}} dz$$

d_1 and d_2 are given by

$$\begin{aligned} d_1 &= \frac{log(\frac{S_0}{X}) + (r - y + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= \frac{log(\frac{S_0}{X}) + (r - y - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \\ &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

with log being the natural logarithm and σ being the forward volatility (from t to T) per year according to the given volatility curve. The analytic solution for the value of a European put option is given by:

$$P_t = Xe^{-r(T-t)}N(-d_2) - S_0 * e^{-y(T-t)} * N(-d_1)$$

- by the use of Adaptive Integration. Here we give a rough description on how Adaptive Integration is used to valuate a vanilla equity option:

  1. build the equity price grid due to the given number of equity price grid points.
  2. determine the value of the option at the option expiry date T. In case of a call option the life option value V(S(i),T) (which is equal to the option value OV(S(i),T) at grid point S(i) is given by

     $$OV(S(i), t_1) = max(0, S(i) - X)$$

  3. propagate back in time due to the given maximal length of a time step (i.e. building the grid in time direction) - we call it maxdt (the default value in the Mathematica Front End is 30 days). Let us assume we have got the values at date t_2. The next date during the calculation is (t_2 - maxdt) unless there is a key date between (t_2 - maxdt) and t_2. A key date is a dividend date, a Bermudan exercise date, the valuation day or one day after the valuation date. If one or more key dates are between (t_2 - maxdt) and t_2, the latest of these key dates is the next considered date. Let us call this date t_1.
     The life value $V(S(i), t_1)$ is obtained by

     $$V(S(i), t_1) = e^{-r(t_2 - t_1)} \int_{-\infty}^{\infty} P[\xi] * OV(S(i) + \xi - D(t_2), t_2) d\xi$$

     where $P[\xi]$ is the probability density (in the risk-free world) that S(i) (given at t_1) moves to S(i)+ξ at t_2. Under the assumption of the geometric Brownian motion for the underlying equity price this density is the density of a lognormal distribution, r, y and σ are given by the forward values from t_1 to t_2 according to the corresponding yield and volatility curves. $D(t_2)$ is the amount of the discrete dividend if t_2 is a dividend date and 0 otherwise.
     If the option is not exercisable at date t_1 (i.e. if the option is European or Bermudan with t_1 not being part of the given Bermudan exercise schedule), the option value $OV(S(i), t_1)$ is given as

     $$OV(S(i), t_1) = V(S(i), t_1)$$

If the option is exercisable at date t_1 (i.e. if the option is American or Bermudan with t_1 being part of the given Bermudan exercise schedule), the option value $OV(S(i), t_1)$ is given as

$$OV(S(i), t_1) = max(V(S(i), t_1), S(i) - X)$$

in case of a call option and

$$OV(S(i), t_1) = max(V(S(i), t_1), X - S(i))$$

in case of a put option.

4. the propagation backwards in time due to step 3) is performed until the valuation date t is reached. The option value for equity price S is given as

$$OV(S, t) = V(S, t)$$

if the option is not exercisable at the valuation date.
If the option is exercisable at the valuation date the option value due to equity price S is given as

$$OV(S, t) = max(V(S, t), S - X)$$

in case of a call option and

$$OV(S, t) = max(V(S, t), X - S)$$

in case of a put option. The values for delta, gamma and theta can easily be calculated during the same calculation procedure. For the calculation of vega, the volatility convexity and delta vega, the given volatility curve is shifted by $\pm 1\%$ (parallel shifts) and the value and the delta for these moved volatilities are calculated. If the vega is not needed it should not be calculated to save computation time.

1.3 Vanilla Equity Option under local volatilities

Vanilla Equity Options under a local volatilty surface can be valuated by the use Adaptive Integration like in the Black Scholes case. The only difference is, that the volatility σ in a point (S,t) is taken from the local volatility surface.

1.4 Vanilla Equity Option under Heston Model

By means of the risk neutral valuation formula the price of any option can be written as an expectation of the discounted payoff of this option.

$$v(x, t_0) = e^{-r(\Delta t)} E^Q[v(y, T)|x] = e^{-r\Delta t} \int_R v(y, T) f(y|x) dy \qquad (1)$$

where v denotes the option value, Δt is the difference between the initial date t_0 and maturity T, $E^Q[]$ is the expectation operator under risk neutral measure Q, x and y are the state variables at time t_0 and T, $f(x|y)$ is the probability density of y given x and r is the risk neutral interest rate.
Since the probability density function which appears in the integration in the

original pricing domain is not known explicitely, its Fourier transform, the characteristic function - which is avaliable in the Heston model - is used. Therefore the problem of option pricing is transformed to the Fourier domain, where the Fourier transformed integrals can be solved efficiently. In our approach we use a Fourier cosine expansion in the context of numerical integration als an alternative for the methods based on FFT, which further improves the speed of pricing plain vanilla options. The characteristic function of the log-asset price in the Heston model is given by

$$\Phi(\omega) = exp\{\frac{\theta\kappa}{\sigma^2}((\kappa-\rho\sigma\omega i-d)T-2ln(\frac{1-ge^{-dT}}{1-g})+\frac{v_0}{\sigma^2}(\kappa-\rho\sigma\omega i-d)\frac{1-e^{-dT}}{1-ge^{-dT}}\}$$

$$d = ((\rho\sigma\omega i-\kappa)^2 - \sigma^2(-i\omega-\omega^2))^{\frac{1}{2}},$$
$$g = \frac{\kappa-\rho\sigma\omega i-d}{\kappa-\rho\sigma\omega i+d}$$

1.4.1 The COS-method

Numerical integration methods have to solve certain forward or inverse Fourier integrals. The density and its characteristic function, $f(x)$ and $\phi(x)$, form an example of a Fourier pair.

$$\phi(\omega) = \int_R e^{ix\omega} f(x)dx \tag{2}$$

$$f(x) = \int_R e^{-ix\omega}\phi(\omega)dx \tag{3}$$

The main idea of the COS methodology is to reconstuct the whole integral in (3) from its Fourier-cosine series expansion, extracting the series coefficients directly from the integrand. The cosine expansion of a function f with support $[0, \pi]$ is given by

$$f(\theta) = \sum_{k=0}^{\infty}{}' A_k cos(k\theta) \quad \text{with} \quad A_k = \frac{2}{\pi}\int_0^{2\pi} f(\theta)cos(k\theta)d\theta$$

where $\sum'$ denotes that the first term in the summation is weighted by 0.5. For the treatment of functions with any other support, a change of variables is required.
Since any L_2 function has a cosine expansion, when it is finitely supported, and the integrands in (3) decay at $\pm\infty$ we can truncate the integration range to a finite interval [a,b] without losing accurancy. We additionaly make use of the fact that a density function tends to be smooth and therefore only a few terms in the expansion lead to a good approximation.

Following the ideas of Fang & Oosterlee (2008) the following COS-formula for a general underlying process holds

$$v(x,t_0) \approx e^{-r\Delta t}\sum_{k=0}^{N-1}{}' Re\{\phi(\frac{k\pi}{b-a};x)e^{-ik\pi\frac{a}{b-a}}\}V_k \tag{4}$$

with

$$V_k = \frac{2}{b-a}\int_a^b v(y,T) cos(k\pi \frac{y-a}{b-a}) dy \quad (5)$$

Denoting the log asset prices by

$$x = ln(\frac{S_0}{K}) \qquad y = ln(\frac{S_T}{K})$$

with S_t being the underlying price at time t and K being the strike price the payoff for european options reads

$$v(y,T) = [\alpha K(e^y - 1)]^+ \qquad \text{with} \quad \alpha = \begin{cases} 1 & \text{if call option} \\ -1 & \text{if put option} \end{cases}$$

The pricing formula can be simplified for the Heston model because the coefficients V_k can be obtained analytically for plain vanilla options:

$$\begin{aligned} V_k^{call} &= \frac{2}{b-a}\int_0^b K(e^y - 1) cos(k\pi \frac{y-a}{b-a}) dy = \frac{2}{b-a}(\chi(0,b) - \psi(0,b)) \\ V_k^{put} &= \frac{2}{b-a}(-\chi(a,0) + \psi(a,0)) \end{aligned}$$

with

$$\begin{aligned} \chi_k(c,d) &:= \frac{1}{1+(\frac{k\pi}{b-a})^2}[cos(k\pi\frac{d-a}{b-a})e^d - cos(k\pi\frac{c-a}{b-a})e^c \\ &+ \frac{k\pi}{b-a} sin(k\pi\frac{d-a}{b-a})e^d - \frac{k\pi}{b-a} sin(k\pi\frac{c-a}{b-a})^c] \\ \psi_k(c,d) &:= \begin{cases} [sin(k\pi\frac{d-a}{b-a}) - sin(k\pi\frac{c-a}{b-a})]\frac{b-a}{k\pi} & k \neq 0 \\ d-c & k = 0 \end{cases} \end{aligned}$$

2 Vanilla FX Option

2.1 Introduction

An FX object represents the foreign exchange (FX) rate between 2 given currencies. An FX object is given by a spot rate (the FX rate at the considered date) (e.g. 1.55), a given foreign currency (e.g. GBP) and a domestic currency (e.g. EUR). The spot FX rate is the price for one unit of the foreign currency, expressed in units of domestic currency. In the risk-free world, the time development of the FX rate depends on the interest rate curves in the foreign and in the domestic currency.
A vanilla FX (foreign exchange) option is a financial instrument with the following properties: The owner of the option has the right (not the obligation) to buy (in case of a call option) or sell (in case of a put option) 1 unit of the foreign currency at the option expiry date (in case of a European option), at certain dates (in case of a Bermudan option) or every day (in case of an American option) for a certain amount (the strike FX rate) of foreign currency. The underlying FX has a given spot FX rate. One is interested in the fair value

of such options at the valuation date (a possible trade will be settled at the settlement date) due to a given interest rate (yield curve or swap curve) and according to a given volatility curve which describes the random behavior of the underlying FX rate. For hedging purposes, the Greeks are of interest: delta, gamma (first and second derivative of the option value with respect to the spot FX rate of the underlying currency), theta (first derivative of the option value with respect to time), vega and volatility convexity (first and second derivative of the option value with respect to volatility), delta vega (mixed derivative with respect to the spot FX rate of the underlying currency and with respect to the volatility).

2.2 Vanilla FX Option under Black Scholes Model

By the use of the UnRisk PRICING ENGINE the value of a vanilla FX option may be calculated in two ways:

- by the use of the analytic formula (i.e. the Black-Scholes formula) which is valid for the valuation of a European option on an FX.
  The analytic solution for the value of a European call option is given by:

  $$C_t = F * e^{-r_f(T-t)} * N(d_1) - Xe^{-r(T-t)}N(d_2)$$

  where F is the given fx spot price of the equity, X is the strike price of the option, r_f is the continuous forward rate (from t to T) per year due to the given domestic yield curve, r is the continuous forward rate (from t to T) per year according to the given yield curve, T is the expiry date of the option, t is the valuation date (T-t has to be expressed in years), N(x) is the cumulative probability distribution for the standard normal distribution at x, i.e.

  $$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}} dz$$

  d_1 and d_2 are given by

  $$\begin{aligned} d_1 &= \frac{log(\frac{F}{X}) + (r - r_f + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= \frac{log(\frac{F}{X}) + (r - r_f - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \\ &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

  with log being the natural logarithm and σ being the forward volatility (from t to T) per year according to the given volatility curve. The analytic solution for the value of a European put option is given by:

  $$P_t = Xe^{-r_d(T-t)}N(-d_2) - F * e^{-r_f(T-t)} * N(-d_1)$$

- by the use of Adaptive Integration.
  Here the steps are the same as for vanilla equity options with the foreign interest rate replacing the continuous equity yield, the domestic interest

rate replacing the interest rate, the spot FX rate replacing the spot price of the equity and the strike FX rate of the vanilla FX option replacing the strike price of the vanilla equity option. Discrete dividends do not occur in FX options.

2.3 Vanilla FX Option under local volatility

Vanilla FX Options under a local volatilty surface can be valuated by the use Adaptive Integration in the same way as in the Black Scholes case. We only need to replace the dividend yiels by the foreign interest rate and take the volatility σ in a point (S,t) from the local volatility surface.

3 Equity Barrier Option

3.1 Introduction

In section 1, we have dealt with vanilla equity options. This section is devoted to equity barrier options. Barrier options become worthless if a given barrier is passed by the equity price (out-options), or they start to exist when a given barrier is passed (in-options).
To be more specific, in case of a knock-out option, the option stays alive as long as the equity price is lower (in case of an up&out option) or higher (in case of a down&out option) than a certain evel (the barrier). In case of a knock-in option, the option starts to exist as the equity price reaches a certain level from below (in case of an up&in option) or from above (in case of a down&in option). If the option is knocked out (in case of a knock-out option) or never knocked in (in case of a knock-in option) the investor may get a certain fixed amount of money, the so called rebate, which may be paid as soon as the knock-out occurs (immediately) or at the expiry date (deferred). One is interested in the fair value of such an option at the valuation date (a possible trade will be settled at the settlement date) due to given interest rates (yield curve or swap curve) and due to a given volatility curve which describes the random behavior of the underlying equity.

3.2 Equity Barrier Option under Black Scholes model

By the use of the UnRisk PRICING ENGINE the value of an equity barrier option may be calculated in two ways:

- by the use of the analytic solution which is valid for the valuation of a European option on a non-dividend paying equity, provided the yield curve and the volatility curve are flat. If a rebate shall be taken into account by the analytic formula , it must be paid immediately after the knock-out. Let A, B, C, D, E, F be defined as follows (η and ϕ are 1 or

-1, depending on the option types, see below):

$$
\begin{aligned}
A &= \phi S e^{-y(T-t)} N(\phi x_1) - \phi X e^{-r(T-t)} N(\phi x_1 - \phi\sigma\sqrt{T-t}) \\
B &= \phi S e^{-y(T-t)} N(\phi x_2) - \phi X e^{-r(T-t)} N(\phi x_2 - \phi\sigma\sqrt{T-t}) \\
C &= \phi S e^{-y(T-t)} (\frac{H}{S})^{2(\mu+1)} N(\eta y_1) - \phi X e^{-r(T-t)} (\frac{H}{S})^{2\mu} N(\eta y_1 - \eta\sigma\sqrt{T-t}) \\
D &= \phi S e^{-y(T-t)} (\frac{H}{S})^{2(\mu+1)} N(\eta y_2) - \phi X e^{-r(T-t)} (\frac{H}{S})^{2\mu} N(\eta y_2 - \eta\sigma\sqrt{T-t}) \\
E &= K e^{-r(T-t)} [N(\eta x_2 - \eta\sigma\sqrt{T-t}) - (\frac{H}{S})^{2\mu}] N(\eta y_2 - \eta\sigma\sqrt{T-t}) \\
F &= K[(\frac{H}{S})^{\mu+\lambda} N(\eta z) + (\frac{H}{S})^{\mu-\lambda} N(\eta z - 2\eta\lambda\sigma\sqrt{T-t})]
\end{aligned}
$$

where S is the spot price of the equity, X is the strike price of the option, H is the barrier level, K is the rebate, r is the continuous interest rate (constant), y is the continuous equity yield (constant), σ is the volatility of the underlying equity (constant), T is the expiry date, t is the valuation date and N(x) is the cumulative probability distribution for the standard normal distribution at x, i.e.

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}} dz$$

The other quantities are given by

$$
\begin{aligned}
x_1 &= \frac{ln(\frac{S}{X})}{\sigma\sqrt{T-t}} + (1+\mu)\sigma\sqrt{T-t} \\
x_2 &= \frac{ln(\frac{S}{H})}{\sigma\sqrt{T-t}} + (1+\mu)\sigma\sqrt{T-t} \\
y_1 &= \frac{ln(\frac{H^2}{SX})}{\sigma\sqrt{T-t}} + (1+\mu)\sigma\sqrt{T-t} \\
y_2 &= \frac{ln(\frac{H}{S})}{\sigma\sqrt{T-t}} + (1+\mu)\sigma\sqrt{T-t} \\
z &= \frac{ln(\frac{H}{S})}{\sigma\sqrt{T-t}} + \lambda\sigma\sqrt{T-t} \\
\mu &= \frac{r - y - \frac{\sigma^2}{2}}{\sigma^2} \\
\lambda &= \sqrt{\mu^2 + \frac{2r}{\sigma^2}}
\end{aligned}
$$

The values of the different equity barrier options may then be calculated as (see, e.g., [Haug])

Up-and-out call with S < H:

$$
\begin{aligned}
&C(\text{up\&out}, X > H) = F, && \eta = -1, \phi = 1 \\
&C(\text{up\&out}, X < H) = A - B + C - D + F, && \eta = -1, \phi = 1
\end{aligned}
$$

Down-and-out call with S > H:

$$C(\text{down\&out}, X > H) = A - C + F, \qquad \eta = 1, \phi = 1$$
$$C(\text{down\&out}, X < H) = B - D + F, \qquad \eta = 1, \phi = 1$$

Up-and-out Put with S < H:

$$P(\text{up\&out,X>H}) = B - D + F, \qquad \eta = -1, \phi = -1$$
$$P(\text{up\&out,X<H}) = A - C + F, \qquad \eta = -1, \phi = -1$$

Down-and-out put with S > H:

$$P(\text{down\&out,X>H}) = A - B + C - D + F, \qquad \eta = 1, \phi = -1$$
$$P(\text{down\&out,X<H}) = F, \qquad \eta = 1, \phi = -1$$

Up-and-in call with S < H:

$$C(\text{up\&in,X>H}) = A + E, \qquad \eta = -1, \phi = 1$$
$$C(\text{up\&in,X<H}) = A - C + D + E, \qquad \eta = -1, \phi = 1$$

Down-and-in call with S > H:

$$C(\text{down\&in,X>H}) = C + E, \qquad \eta = 1, \phi = 1$$
$$C(\text{down\&in,X<H}) = A - B + D + E, \qquad \eta = 1, \phi = 1$$

Up-and-in put with S < H:

$$P(\text{up\&in,X>H}) = A - B + D + E, \qquad \eta = -1, \phi = -1$$
$$P(\text{up\&in,X<H}) = C + E, \qquad \eta = -1, \phi = -1$$

Down-and-in put with S > H:

$$P(\text{down\&in,X>H}) = B - C + D + E, \qquad \eta = 1, \phi = -1$$
$$P(\text{down\&in,X<H}) = A + E, \qquad \eta = 1, \phi = -1$$

- by the use of Adaptive Integration
  The steps are the same as for vanilla equity options, however the transition probability (in the risk-free measure) must be split into two parts: If $p(s,\xi)$ is the probability density of S (at t_1) moving to S+ξ at t_2 (both assumed to be below the barrier in the case of an up-and-out option), then

$$p(S,\xi) = p_1(S,\xi) + p_2(S,\xi)$$

  with p_1 being the density of S moving to $S+\xi-D(t_2)$ without hitting the barrier and p_2 being the density of S going to $S+\xi-D(t_2)$ with hitting the barrier.
  For $S(i) < H$ the life value is then obtained by

$$\begin{aligned} V(S(i),t_1) &= e^{-r(T-t)}[\textstyle\int_{-\infty}^{H-S(i)} p_1(S(i),\xi) * OV(S(i)+\xi-D(t_2),t_2)d\xi \\ &+ \textstyle\int_{-\infty}^{H-S(i)} p_2(S(i),\xi) * \kappa d\xi + \int_{H-S(i)}^{\infty} p(S(i),\xi)\kappa d\xi] \end{aligned}$$

  For $S(i) \geq H$ the life value is then obtained by

$$V(S(i),t_1) = \kappa$$

  with κ is the rebate if the rebate is paid immediately, or the discounted rebate (from expiry to t_1) if the rebate is paid at the expiry date.
  Some remarks:

  1. For the deviation of p_1, the reflection principle has to be applied. For details see, e.g., Albrecher et al. (2009).
  2. In the implementation of Adaptive Integration, the rebate (if any) is paid only at dates hit by the timestepping scheme.
  3. In the densities p_1 and p_2 flat forward rates and flat forward volatilities (from t_1 to t_2) are used. If the maximal length of the time step decreases, the actual curves (yield and volatility) are matched better. See below for an example on the influence of the time step.
  4. There are some additional thoughts for knock-in options: Assume we have an up-and-in option and a price S below the barrier. Then there are two option values at S, the value of the option not being knocked in yet, and the value of the option having already been knocked in (this is the vanilla value). As soon as the two option values are introduced, the rest can be done in a similar way to the knock-out option.

3.3 Equity Barrier Option under local volatility

Equity Barrier Options under a local volatilty surface can be valuated by the use Adaptive Integration like in the Black Scholes case. The only difference is, that the volatility σ in a point (S,t) is taken from the local volatility surface.

3.4 Equity Barrier Option under Heston Model

In contrast to the Black Scholes model where we have analytic formulae for the prices of barrier options, in the Heston model we have to use numerical methods to price these options. In the UnRisk PRICING ENGINE we approximate the stochastic differential equation using a discrete Euler sheme. To generate correlated random variables we choose Z_1 and Z_2 independently from a standard normal distribution N(0,1) and calculate ΔW_t^1 and ΔW_t^2 for a given time step Δt as

$$\Delta W_t^1 = Z_1\sqrt{\Delta t}$$
$$\Delta W_t^1 = (\rho Z_1 + \sqrt{1-\rho^2}Z_2)\sqrt{\Delta t}$$

Starting from S_0 and v_0 the discrete paths of the Monte Carlo simulation can be calculated iteratively using

$$S_i = S_{i-1} + (r_d - r_f)S_{i-1}\Delta t + \sqrt{v_{i-1}}S_{i-1}\Delta W_t^1$$
$$v_i = v_{i-1} + (\kappa(\theta - v_{t-1})\Delta t + \sigma\sqrt{v_{i-1}}\Delta W_t^2$$

where $S_i = S(i * \Delta t)$ and $v_i = v(i * \Delta t)$
When we consider continously monitored barrier options, the hitting time error using a discrete Monte Carlo method is given by $O(1/\sqrt{N})$, if we use N time steps. The convergence is rather slow, because the exact path of the underlying asset may hit the barrier between the discrete time points of the discrete Monte Carlo sheme. To reduce the hitting time error near the barrier, we use a random variable which takes the conditional probabilty of hitting the barrier between two discrete time points into account.
The payoffs of discretely monitored Barrier option with strike K, constant Barrier H and maturity T can be written as

$$\begin{aligned}
\text{up and out:} &\quad (\Phi(S_T - K))^+ 1_{max_{i\in\{1,\dots,n\}} S_{t_i} < H},\\
\text{up and in:} &\quad (\Phi(S_T - K))^+ 1_{max_{i\in\{1,\dots,n\}} S_{t_i} > H},\\
\text{down and out:} &\quad (\Phi(S_T - K))^+ 1_{min_{i\in\{1,\dots,n\}} S_{t_i} > H},\\
\text{down and in:} &\quad (\Phi(S_T - K))^+ 1_{min_{i\in\{1,\dots,n\}} S_{t_i} < H},
\end{aligned}$$

where $\Phi = \pm 1$ determines if the option is a call(+1) or a put(-1) and t_i with $0 < t_1 < ... < t_n = T$ are the barrier monitoring dates. Since the barrier option value for a path in the Monte Carlo simulation becomes worthless, if the price of the underlying stock exeeds the Barrier level before time T, we only consider paths in our simulation, which stay in the allowed region.

4 FX Barrier Option

FX Barrier Options can be valuated by the UnRisk PRICING ENGINE in two ways:

- by the use of the analytic solution which is valid only for European FX barrier options and flat (domestic, foreign) interest rates and flat volatilities. It is also just valid if the rebate is paid as soon as the option is knocked-out (i.e. as soon as the FX rate moves across the barrier level).

The closed form solutions are obtained by replacing in the formulas for equity barrier options, the continuous equity yield by the foreign interest rate and the spot price of the equity by the spot FX rate.

- by the use of Adaptive Integration.
  Again, in the algorithm for equity barrier options, the continuous equity yield is replaced by the foreign interest rate. When Adaptive Integration is used, the interest rates (domestic, foreign) and the volatility curve need not to be flat any more.

5 Interpretation of the Greeks

5.1 Greeks for Equity Derivatives

The UnRisk PRICING ENGINE uses the following scaling and units:

- **Value:** in currency units
- **Delta:** unit 1. If the spot price changes by dS, the Value changes by $\text{Delta} * dS$ (first order approximation)
- **Gamma:** unit $\frac{1}{currency}$. If the spot price changes by dS, then the value changes by $\text{Delta} * dS + \frac{1}{2}\text{Gamma} * (dS)^2$ (second order approximation)
- **Theta:** Change of the option value if the spot price remains unchanged and the valuation and settlement date are shifted by one business day.
- **Vega:** Change of the option value if the volatility curve is shifted by one percent.
- **Volatility Convexity:** Second order change if the volatility is shifted by one percent: $V(\sigma + d\sigma) \approx V(\sigma) + \text{Vega} * d\sigma + \frac{1}{2}\text{Volatility Convexity} * (d\sigma)^2$
- **Delta Vega:** Change of Delta if the volatility is shifted by one percent.

5.2 Greeks for FX Derivatives

The UnRisk PRICING ENGINE uses the following scaling and units:

- **Value:** in currency units
- **Delta:** unit 1. If the spot fx price changes by dF, the Value changes by $\text{Delta} * dF$ (first order approximation)
- **Gamma:** unit $\frac{1}{currency}$. If the spot price changes by dF, then the value changes by $\text{Delta} * dF + \frac{1}{2}\text{Gamma} * (dF)^2$ (second order approximation)
- **Theta:** Change of the option value if the spot price remains unchanged and the valuation and settlement date are shifted by one business day.
- **Vega:** Change of the option value if the volatility curve is shifted by one percent.

- **Volatility Convexity:** Second order change if the volatility is shifted by one percent: $V(\sigma + d\sigma) \approx V(\sigma) + \text{Vega} * d\sigma + \frac{1}{2}\text{Volatility Convexity} * (d\sigma)^2$
- **Delta Vega:** Change of Delta if the volatility is shifted by one percent.

References

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