Technical Paper:

Valuation of Interest Rate Instruments

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1 One Factor Interest Rate Model

Within the UnRisk PRICING ENGINE, the value of interest rate instruments under a 1 factor interest rate model is calculated by the use of Adaptive Integration. In the following we give a rough description of the algorithm for a callable / putable fixed rate bond (we neglect the incorporated call / put notice feature in order to keep the formulation as understandable as possible):

1. build the spot rate grid due to the given number of spot rate grid points.

2. at the maturity date T the dirty value of the callable / putable fixed rate bond at a spot rate $r(i)$ is given as the dirty value of the underlying fixed rate bond (callability and putability are assumed not to be allowed at the maturity date)
$$CPV(r(i), T) = R + CF(T)$$
where R is the final redemption of the bond and $CF(T)$ is the cashflow at the maturity date (0 if there is no coupon at maturity).

3. propagate back in time due to the given maximal length of a time step (i.e. building the grid in time direction) - we call it maxdt (the default value in the Mathematica Front End is 30 days). Let us assume we have got the dirty callable / putable bond values at date $t2$. The next considered date $t1$ ($t1 < t2$) is given by max(date maxdt days before $t2$, coupon date before $t2$, call date before $t2$, put date before $t2$, valuation date, settlement date, in case of a General Hull & White model: a date at which one of the model parameters changes). Therefore it is possible to hit all key dates - coupon dates, call dates, put dates, valuation date, settlement date, in case of a General Hull & White model: the dates at which the model parameters change. The life value $LBV(r(i), t1)$ of the fixed rate bond (i.e. the value of the fixed rate bond under the assumption that it is not called at t1 and between $t1$ and $t2$) is given as:
$$LBV(r(i), t1) = \int_{-\infty}^{+\infty} DF(r(i), \xi, t1, t2) * P(\xi) CPV(r(i) + \xi, t2) d\xi + CF(t1)$$
where $DF(r(i),\ \xi,\ t1,\ t2)$ is the discount factor form $t2$ to $t1$, provided the interest rate at $t1$ is $r(i)$ and at $t2$ it is $r(i) + \xi$, and $P(\xi)$ is the probability density that $r(i)$ (given at $t1$) moves to $r(i) + \xi$ at $t2$. $CF(t1)$ is the coupon at $t1$ (0 if there is no coupon at $t1$). The dirty value of the callable / putable fixed rate bond is then given as
$$CPV(r(i), t1) = \min((\text{Call+caccrued}), \max(LBV(r(i), t1), (\text{Put+paccrued})))$$
if the fixed rate bond is callable and putable at $t1$. caccrued is the accrued interest at $t1$ if the call accrued switch is set to True, 0 otherwise. paccrued is the accrued interest at $t1$ if the put accrued switch is set to True, 0 otherwise.
$$CPV(r(i), t1) = \max(LBV(r(i), t1), (\text{Put} + \text{paccrued}))$$
if the fixed rate bond is just putable at $t1$.
$$CPV(r(i), t1) = \min((\text{Call} + \text{caccrued}), LBV(r(i), t1))$$

if the fixed rate bond is just callable at $t1$.

$$CPV(r(i), t1) = LBV(r(i), t1)$$

if the fixed rate bond is neither callable nor putable at $t1$.

4. propagate backwards in time until the settlement date ts is reached. Between settlement date and valuation date we proceed as above but without discounting in the integral.

   The option value $OV(r,t)$ is given by ($BV(r(i),t)$ is the dirty value of the underlying fixed rate bond)

$$OV(r,t) = CPV(r,t) - BV(r,t)$$

Note that, e.g., for a bond which is callable but not putable the option value is negative (as it is seen from the investor's point of view).

2 Hull & White 2 Factor Model

The value of interest rate instruments under a Hull & White 2 factor model is calculated by the use of Finite Elements with Streamline Diffusion. In the following the valuation algorithm for a callable / putable general steepener is presented:

Step 1: Determination of the time discretization: Based on the maximal time step and on the given key dates the time discretization is determined in a way such that the actual time step does not exceed the maximal time step maxdt (the default value in the Mathematica Front End is 20 days) and all key dates are hit. Key dates can be coupon set dates, coupon dates, call dates, put dates, the settlement date, the valuation date, or a date at which one of the model parameters changes.

Step 2: Determination of the space discretization: Depending on the volatilities of the two factors and on the lifetime of the considered instrument the size of the discretization grid is determined as it is described in the paper "Numerical Methods in UnRisk" in section "Streamline Diffusion" (each direction represents one factor). The discretization grid itself consists of rectangles. The number of rectangles in each direction does not exceed the maximal number given by the function call option NumericalParameters2D, but it may be less if the required accuracy is obtained.

Step 3: Determination of starting values: Calculate the dirty value of the callable / putable general steepener $CPV(r,u,T)$ at maturity date T in a grid point ($r(i)$, $u(i)$). The steepener is assumed neither to be callable nor to be putable at the maturity date, therefore, the dirty value of the callable / putable steepener $CPV(r,u,T)$ is given by the dirty value of the underlying steepener $SV(r,u,T)$:

$$CPV(r(i), u(i), T) = SV(r(i), u(i), T)$$

The dirty value of the underlying steepener is given by:

$$SV(r(i), u(i), T) = R + CF(r(i), u(i), T)$$

where R is the final redemption of the steepener and $CF(r(i), u(i), T)$ is

- an already known cashflow (if T is a coupon date and the corresponding set date is before the valuation date)
- a cashflow defined by the specified coupon (if T is a coupon set date)
- 0 otherwise.

Step 4 is repeated ($t_{j+1} = t_j$) until the settlement date is reached!

Step 4: Propagate back in time: Under the assumption that we know the dirty value $CPV^(j+1)$ at time t_{j+1} (starting with $t_{j+1} = T$), we want to determine the value at time t_j. The time step $\Delta t^j = t_{j+1} - t_j$ is given by the time discretization determined at the beginning of the algorithm. To calculate the value at time t_j means to solve the following partial differential equation for V^j using the method of Finite Elements including Streamline Diffusion as upwind technique:

$$
\begin{aligned}
&\frac{CPV^{j+1} - V^j}{\Delta t^j} + \\
&\alpha\Bigg(\frac{1}{2} * \sigma_1^2(t_{j+1})\frac{\partial^2 CPV^{j+1}}{\partial r^2} + \rho(t_{j+1})\sigma_1(t_{j+1})\sigma_2(t_{j+1})\frac{\partial^2 CPV^{j+1}}{\partial r \partial u} + \\
&\frac{1}{2} * \sigma_2^2(t_{j+1})\frac{\partial^2 CPV^{j+1}}{\partial u^2} + (\theta(t_{j+1}) + u - a(t_{j+1})r)\frac{\partial CPV^{j+1}}{\partial r} - \\
&-b(t_{j+1})u\frac{\partial CPV^{j+1}}{\partial u} - rCPV^{j+1}\Bigg) + \\
&(1-\alpha)\Bigg(\frac{1}{2} * \sigma_1^2(t_j)\frac{\partial^2 V^j}{\partial r^2} + \rho(t_j)\sigma_1(t_j)\sigma_2(t_j)\frac{\partial^2 V^j}{\partial r \partial u} + \\
&\frac{1}{2} * \sigma_2^2(t_j)\frac{\partial^2 V^j}{\partial u^2} + (\theta(t_j) + u - a(t_j)r)\frac{\partial V^j}{\partial r} - \\
&b(t_j)u\frac{\partial V^j}{\partial u} - rV^j\Bigg) = 0
\end{aligned}
$$

The life value of the steepener at time t_j in $(r(i), u(i))$ (i.e., the value of the steepener under the assumption that it is not called at t_j and not called between t_j and t_{j+1}) is then given by:

$$(LV^j)(r(i), u(i)) = (V^j)(r(i), u(i)) + (CF^j)(r(i), u(i)),$$

where $(CF^j)(r(i), u(i))$ is

- an already known cashflow (if t_j is a coupon date and the corresponding set date is before the valuation date)
- a cashflow defined by the specified coupon (if t_j is a coupon set date)
- 0 otherwise.

The dirty value of the callable / putable steepener is given by:

$$(CPV^j)(r(i), u(i)) = \min(\text{Call}, \max((LV^j)(r(i), u(i)), \text{Put}))$$

if the steepener is callable and putable at t_j,

$$(CPV^j)(r(i),u(i)) = \max((LV^j)(r(i),u(i)), \mathrm{Put})$$

if the steepener is just putable at t_j,

$$(CPV^j)(r(i),u(i)) = \min(\mathrm{Call}, (LV^j)(r(i),u(i)))$$

if the steepener is just callable at t_j,

$$(CPV^j)(r(i),u(i)) = (LV^j)(r(i),u(i))$$

if the steepener is neither callable nor putable at t_j.

Step 4 is repeated ($t_{j+1} = t_j$) until the settlement date is reached!

Step 5: We know the rates r and $u(=0)$ at the valuation date t, but we want to know the dirty value of the bond at the settlement date. In order to obtain this value we have to solve the following problem between the settlement date and the valuation date:

$$\begin{aligned}
&\frac{CPV^{j+1} - V^j}{\Delta t^j} + \\
&\alpha\Bigg(\frac{1}{2} * \sigma_1^2(t_{j+1})\frac{\partial^2 CPV^{j+1}}{\partial r^2} + \rho(t_{j+1})\sigma_1(t_{j+1})\sigma_2(t_{j+1})\frac{\partial^2 CPV^{j+1}}{\partial r \partial u} + \\
&\frac{1}{2} * \sigma_2^2(t_{j+1})\frac{\partial^2 CPV^{j+1}}{\partial u^2} + (\theta(t_{j+1}) + u - a(t_{j+1})r)\frac{\partial CPV^{j+1}}{\partial r} - \\
&-b(t_{j+1})u\frac{\partial CPV^{j+1}}{\partial u}\Bigg) + \\
&(1-\alpha) * \Bigg(\frac{1}{2} * \sigma_1^2(t_j)\frac{\partial^2 V^j}{\partial r^2} + \rho(t_j)\sigma_1(t_j)\sigma_2(t_j)\frac{\partial^2 V^j}{\partial r \partial u} + \\
&\frac{1}{2} * \sigma_2^2(t_j)\frac{\partial^2 V^j}{\partial u^2} + (\theta(t_j) + u - a(t_j)r)\frac{\partial V^j}{\partial r} - \\
&b(t_j)u\frac{\partial V^j}{\partial u}\Bigg) = 0
\end{aligned}$$

This is in principle the same problem as in Step 4, but without discounting. The dirty value of the callable / putable general steepener $(CPV^j)(r(i),u(i))$ is then calculated as described in Step 4.

The option value $OV(r,t)$ is given by

$$OV(r,u,t) = CPV(r,u,t) - SV(r,u,t).$$

$SV(r,u,t)$ is the dirty value of the underlying general steepener.

3 Two 1 Factor Interest Rate Models

The value of interest rate instruments under two General Hull & White interest rate models is calculated by the use of Finite Elements with Streamline Diffusion. In the following the valuation algorithm for a callable / putable quanto is presented:

Step 1: Determination of the time discretization: Based on the maximal time step and on the given key dates the time discretization is determined in a way such that the actual time step does not exceed the maximal time step maxdt (the default value in the Mathematica Front End is 20 days) and all key dates are hit. Key dates can be coupon set dates, coupon dates, call dates, put dates, the settlement date, the valuation date, or a date at which one of the model parameters changes.

Step 2: Determination of the space discretization: Depending on the volatilities of the two factors and on the lifetime of the considered instrument the size of the discretization grid is determined as it is described in the paper "'Numerical Methods in UnRisk"' in chapter "'Streamline Diffusion"' (each direction represents one factor). The discretization grid itself consists of rectangles. The number of rectangles in each direction does not exceed the maximal number given by the function call option NumericalParameters2D, but it may be less if the required accuracy is obtained.

Step 3: Determination of starting values: Calculate the dirty value $QV(r1, r2, T)$ of the considered quanto at the maturity date T in a grid point $(r1(i),\ r2(i))$. The quanto is assumed neither to be callable nor to be putable at the maturity date, therefore, the dirty value of the callable / putable quanto $CPQV(r1, r2, T)$ is given by:

$$CPQV(r1(i), r2(i), T) = QV(r1(i), r2(i), T)$$

Step 4: Propagate back in time: Under the assumption that we know the dirty value
$CPQV^{j+1}$ at time t_{j+1} (starting with $t_{j+1} = T$), we want to determine the value at time t_j. The time step $\Delta t^j = t_{j+1} - t_j$ is given by the time discretization determined at the beginning of the algorithm. To calculate the value at time t_j means to solve the following partial differential equation for V^j using the method of Finite Elements including Streamline Diffusion as upwind technique:

$$\begin{aligned}
&\frac{CPQV^{j+1} - V^j}{\Delta t^j} + \\
&\alpha\Bigg(\frac{1}{2} * \sigma_1^2(t_{j+1})\frac{\partial^2 CPQV^{j+1}}{\partial r1^2} + \rho t_{j+1}\sigma_1(t_{j+1})\sigma_2(t_{j+1})\frac{\partial^2 CPQV^{j+1}}{\partial r1 \partial r2} + \\
&\frac{1}{2} * \sigma_2^2(t_{j+1})\frac{\partial^2 CPQV^{j+1}}{\partial r2^2} + (\eta_1(t_{j+1}) - \gamma_1(t_{j+1})r1)\frac{\partial CPQV^{j+1}}{\partial r1} + \\
&(\eta_2(t_{j+1}) - \rho_X\sigma_2(t_{j+1})\sigma_X - \gamma_2(t_{j+1})r2)\frac{\partial CPQV^{j+1}}{\partial r2} - r1 CPQV^{j+1}\Bigg) + \\
&(1-\alpha) * \Bigg(\frac{1}{2} * \sigma_1^2(t_j)\frac{\partial^2 V^j}{\partial r1^2} + \rho(t_j)\sigma_1(t_j)\sigma_2(t_j)\frac{\partial^2 V^j}{\partial r1 \partial r2} + \\
&\frac{1}{2} * \sigma_2^2(t_j)\frac{\partial^2 V^j}{\partial r2^2} + (\eta_1(t_j) - \gamma_1(t_j)r1)\frac{\partial V^j}{\partial r1} + \\
&(\eta_2(t_j) - \rho_X\sigma_2(t_j)\sigma_X - \gamma_2(t_j)r2)\frac{\partial V^j}{\partial r2} - r1V^j\Bigg) = 0
\end{aligned}$$

The life value of the quanto (i.e., the value of the quanto under the assumption that it is not called at t_j and not called between t_j and t_{j+1}) at time t_j in $(r1(i),\, r2(i))$ is given by:

$$(LQV^j)(r1(i), r2(i)) = (V^j)(r1(i), r2(i)) + (CF^j)(r1(i), r2(i)),$$

where $CF^j(r1(i), r2(i))$ is

- an already known cashflow (if t_j is a coupon date and the corresponding set date is before the valuation date)
- a cashflow defined by the specified coupon (if t_j is a coupon set date)
- 0 otherwise.

The dirty value of the callable / putable quanto is given by:

$$(CPQV^j)(r1(i), r2(i)) = \min(\text{Call}, \max((LQV^j)(r1(i), r2(i)), \text{Put}))$$

if the quanto is callable and putable at t_j,

$$(CPQV^j)(r1(i), r2(i)) = \max((LQV^j)(r1(i), r2(i)), \text{Put})$$

if the quanto is just putable at t_j,

$$(CPQV^j)(r1(i), r2(i)) = \min(\text{Call}, (LQV^j)(r1(i), r2(i)))$$

if the quanto is just callable at t_j,

$$(CPQV^j)(r1(i), r2(i)) = (LQV^j)(r1(i), r2(i))$$

if the quanto is neither callable nor putable at t_j.

Step 4 is repeated ($t_{j+1} = t_j$) until the settlement date is reached!

Step 5: We know the spot rates $r1$ and $r2$ at the valuation date t, but we want to know the dirty value of the quanto at the settlement date. In order to obtain this value we have to solve the following problem between the settlement date and the valuation date:

$$\begin{aligned}
&\frac{CPQV^{j+1} - V^j}{\Delta t^j} + \\
&\alpha\bigg(\frac{1}{2} * \sigma_1^2(t_{j+1})\frac{\partial^2 CPQV^{j+1}}{\partial r1^2} + \rho t_{j+1}\sigma_1(t_{j+1})\sigma_2(t_{j+1})\frac{\partial^2 CPQV^{j+1}}{\partial r1 \partial r2} + \\
&\frac{1}{2} * \sigma_2^2(t_{j+1})\frac{\partial^2 CPQV^{j+1}}{\partial r2^2} + (\eta_1(t_{j+1}) - \gamma_1(t_{j+1})r1)\frac{\partial CPQV^{j+1}}{\partial r1} + \\
&(\eta_2(t_{j+1}) - \rho_X\sigma_2(t_{j+1})\sigma_X - \gamma_2(t_{j+1})r2)\frac{\partial CPQV^{j+1}}{\partial r2}\bigg) + \\
&(1-\alpha) * \bigg(\frac{1}{2} * \sigma_1^2(t_j)\frac{\partial^2 V^j}{\partial r1^2} + \rho(t_j)\sigma_1(t_j)\sigma_2(t_j)\frac{\partial^2 V^j}{\partial r1 \partial r2} + \\
&\frac{1}{2} * \sigma_2^2(t_j)\frac{\partial^2 V^j}{\partial r2^2} + (\eta_1(t_j) - \gamma_1(t_j)r1)\frac{\partial V^j}{\partial r1} + \\
&(\eta_2(t_j) - \rho_X\sigma_2(t_j)\sigma_X - \gamma_2(t_j)r2)\frac{\partial V^j}{\partial r2}\bigg) = 0
\end{aligned}$$

This is in principle the same problem as in Step 4, but without discounting. The dirty value of the callable / putable quanto $CPQV^j(r1(i), r2(i))$ is then calculated as described in Step 4.

The option value $OV(r1, r2, t)$ is given by:

$$OV(r1, r2, t) = CPQV(r1, r2, t) - QV(r1, r2, t)$$

4 LIBOR Market Model

The value of a financial instrument under a LIBOR market model is calculated by the use of Monte-Carlo method. In the following the valuation algorithm is presented step-by-step:

Step 1: Determination of the time grid: The time grid is the set of all key dates, the valuation as well as the maturity date. Key dates can be coupon set dates, observation dates, swap dates (of all product rates), coupon dates, call dates, put dates and the settlement date (if the date is not equal to the valuation date)

$$T = \{t_0, t_1, \ldots, t_N\} \,\text{with}\, t_0 = \text{valuation date}$$

Step 2: forward LIBOR rates setup: If the last key date is before the maturity date the number of forward rates is equal to the number of key dates, otherwise the number of forward rates is equal to the number of key dates minus one. The first rate expires at t_1 matures at t_2, the second foward rate expires at t_2 and matures at t_3 and so on until the last forward rate expires at t_{N-1} and matures at t_N.

Step 3: Determination of the MC time discretization: Based on the maximal time step and on the given time grid the Monte-Carlo time discretisation (Brigo & Mercurio (2006))

$$T^{MC} := \{t_0, t_{0_1}^{MC}, t_{0_2}^{MC}, \ldots, t_1, t_{1_1}^{MC}, \ldots, t_{m-1}, t_m\} \,\text{with}\, m \leq N$$

is determined in a way so that the actual time step does not exceed the maximal time step maxdt (the default value in the Mathematica Front End is 100 days) and all dates of the time grid until the maturity date are hit.

Step 4: Determination of sigma (σ) and rho (ρ): Depending on the time grid, on the MC time grid and on the model parameters the volatility and correlation matrices are calculated using the volatility function

$$\sigma_k(t) = \psi_k([a(T_{k-1} - t) + b] \exp^{-c(T_{k-1}-t)} + d)$$

and the correlation function

$$\rho_{i,j} = \exp^{-p_1(\exp^{-p_2 \min(i,j)})|i-j|}$$

Step 5: Determination of starting values: For each foward LIBOR rate the current value is used.

Step 6: Monte-Carlo valuation: Using the Euler scheme for discretisation of (2) from the paper "Calibration of Interest Rate Models" (LIBOR market model section) the following n-dimensional process has to be simulated

$$\begin{aligned} d\ln F_k^d(t_{i+1}^{MC}) &= \ln F_k^d(t_i^{MC}) + \\ &\sigma_k(t_i^{MC}) \sum_{j=\beta(t)}^{k} \left(\tau_j \rho_{j,k} \sigma_j(t_j^{MC}) F_j^d(t_j^{MC})\right)(t_{i+1}^{MC} - t_i^{MC}) - \\ &\frac{\sigma_k^2(t_i^{MC})}{2}(t_{i+1}^{MC} - t_i^{MC}) + \frac{\sigma_k^2(t_i^{MC})}{2}(Z_k(t_{i+1}^{MC}) - Z_k(t_i^{MC})) \end{aligned}$$

where $Z_k(t_{i+1}^{MC}) - Z_k(t_i^{MC})$ is $\sqrt{t_{i+1}^{MC} - t_i^{MC}}\mathcal{N}(0, \rho)$ distributed. The independent draws from the multivariate normal distribution are generated using either the MersenneTwister random number generator or Low-discrepancy sequences with the Brownian bridge method (Jaeckel (2002)). In both cases principal component analysis is used to reduce the dimension of driving factors (see Fries, appendix). At each expiry date the dimension of the simulated forward LIBOR rates decreases by one, resulting in a matrix with the form

$$\begin{pmatrix} F_1^d(t_1^{MC}) & F_1^d(t_2^{MC}) & \dots & F_1^d(t_i^{MC}) & & & & & & & \\ F_2^d(t_1^{MC}) & F_2^d(t_2^{MC}) & \dots & F_2^d(t_i^{MC}) & \dots & F_2^d(t_j^{MC}) & & & & & \\ F_3^d(t_1^{MC}) & F_3^d(t_2^{MC}) & \dots & F_3^d(t_i^{MC}) & \dots & F_3^d(t_j^{MC}) & \dots & F_3^d(t_k^{MC}) & & & \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & & \\ F_n^d(t_1^{MC}) & F_n^d(t_2^{MC}) & \dots & F_n^d(t_i^{MC}) & \dots & F_n^d(t_j^{MC}) & \dots & F_n^d(t_k^{MC}) & \dots & F_n^d(t_{M-1}^{MC}) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ F_N^d(t_1^{MC}) & F_N^d(t_2^{MC}) & \dots & F_N^d(t_i^{MC}) & \dots & F_N^d(t_j^{MC}) & \dots & F_N^d(t_k^{MC}) & \dots & F_N^d(t_{M-1}^{MC}) \end{pmatrix}$$

Every column contains all necessary information to compute all product rates (forward and/or swap rates). The numeraire process at time t_m is the product of all LIBOR rates up to time t_{m-1}

$$N(t_m) = \sum_{i=1}^{m-1} F_i^d(t_i)$$

The sum of all discounted coupons (set at the coupon set date, discounted to the settlement date) and the discounted redemption rate is the dirty value of the product without call or put rights:

$$DV = \sum_{i=1}^{m} CV(t_i)/N(t_i) + RR/N(t_m)$$

Step 7: callability / putability: Extending the least square Monte Carlo method for American options (see Longstaff and Schwartz 2002) we compare for every call/put date (and every run) the regression value of discounted future gains (losses) to the regression value of current gains (losses). If the latter is greater than the first regression value we call/put the option, otherwise we do nothing.

References

Brigo, D. & Mercurio, F. (2006), *Interest Rate Models - Theory and Practice: With Smile, Inflation and Credit*, Springer.

Jaeckel, P. (2002), *Monte Carlo Methods in Finance*, Wiley.